Изв.вузов «ПНД», т.3, № 5, 1995

TOOLS FOR ANALYZING OBSERVED CHAOTIC DATA*

Henry D.I. Abarbanel

4. Model Making in Chaos

The last topic we take up in this article is that of making models in chaotic systems for prediction or control of the source of the observed chaotic signal. In a sense this is both the easiest and the hardest task we have discussed in this article. It is the easiest because it is quite simple to make models of the dynamics which very accurately allow one to predict forward in time from any new initial condition close to or on the attractor within the limits of the intrinsic instabilities embodied in the positive Lyapunov exponents. It is also the hardest because there is no guideline as to which of many functional forms to use for the models and what interpretation to place on the parameters in the models from a physical point of view. In this section we make models on the attractor and evaluate them by how well they do in prediction or possibly control. However, another route to model making which must be based in an understanding of the fundamental physics of the problem would be to develop equations of motion for the dynamical system and then compare the output of those equations of motion not by individual orbit $\mathbf{y}(k)$ to the observed orbit, for these must disagree and be essentially uncorrelated from each other again due to the intrinsic instabilities in the dynamics. Instead the comparison is to be made in terms of the statistical quantities such as fractal dimensions and Lyapunov exponents as we have discussed.

As we noted in our introduction, we work here with data as observed on the attractor alone, thus we cannot hope as an algorithmic matter to make models which would have general validity throughout the system state space. For example, if there is a set of initial conditions in that original phase space which leads to other behaviour than what we have observed and analyzed, it is plausible that our models for the motion in that other basin of attraction could be different. It is also quite possible that we would have been lucky enough or insightful enough to have made a model which encompasses both features of the dynamics. When we make models we have to decide from the outset what kind of functional form we are going to use to express what is certainly fundamentally forms of Newton's laws. The impossibility of selecting in any *a priori* way the correct functional form is stressed by Rissanen [43] who also lays out a clear program for extracting from experimental data the most information available.

Our discussion here will limit itself to outlining how one can use the phase space structure we have built up in the $\mathbf{y}(n)$ to provide effective models of the dynamics which

* *Окончание.* Начало см. Изв. вузов «ПНД», т.3, № 2, 3, 4, 1995

allow one to predict the evolution of any newpoint in the phase space within the basin of attraction which has been observed. The basic idea is that since we have seen how points in a neighbourhood evolve into points in the «next» neighbourhood we ought to be able to provide an appropriate interpolation scheme which would allow us to say that any new point would evolve more or less as its neighbourhood was seen to evolve. Now we'll put some flesh on these bones.

4.1. Local Models

The first kind of model we consider is that of local neighbourhood to neighbourhood maps. The idea follows our construction of local maps for purposes of extracting local Jacobian matrices to use in determining Lyapunov exponents. We imagine that we have a certain local functional form for the dynamics $\mathbf{x} \to \mathbf{F}^{(k)}(\mathbf{x})$ in the neighbourhood of the observed point $\mathbf{y}(k)$:

$$\mathbf{F}^{(k)}(\mathbf{x}) = \Sigma_{m=1}^{M} \mathbf{c}(m,k)\, \phi_m(\mathbf{x}), \tag{55}$$

where the functions $\phi_m(\mathbf{x})$ are a basis set we choose from intuition or good guessing or convenience. These $\phi_m(\mathbf{x})$ could be polynomials or other functions with some appeal. The discussion of what functions to use and how many to use takes us into the difficult subject of multidimensional interpolation [44] which we will not pretend to present here. In an intuitive sense we can say that if we have enough data, then local polynomial approximations to the dynamics is sure to provide accurate local maps. When data becomes sparse or dimensions become high and the number of coefficients in the polynomials correspondingly large, other interpolation functions will probably be more efficient and accurate. I do not have a favorite all purpose basis set to offer or suggest.

Returning to the general problem, which we will then illustrate with polynomials. We go to a particular point $\mathbf{y}(k)$ in the embedding space of dimension d_E and using a device such as the principal component decomposition discussed in the context of local false nearest neighbors select out a d_L dimensional subspace in which to make a model. All distances are evaluated in the $d_E \geq d_L$ dimensional space, but all other computations are done on the d_L dimensional model which takes the selected d_L components of $\mathbf{y}(k)$ into the same components of $\mathbf{y}(k+1)$ via

$$\mathbf{y}(k+1) = \mathbf{F}^{(k)}(\mathbf{y}(k)) = \Sigma_{m=1}^{M} \mathbf{c}(m,k)\, \phi_m(\mathbf{y}(k)). \tag{56}$$

To determine the coefficients in the model we select the N_B nearest neighbors of the phase space point $\mathbf{y}^{(r)}(k)$; $r=1,2,\ldots,N_B$, and minimize

$$\Sigma_{r=1}^{N_B} |\mathbf{y}(r,k+1) - \Sigma_{m=1}^{M} \mathbf{c}(m,k)\, \phi_m(\mathbf{y}^{(r)}(k))|^2. \tag{57}$$

This is a linear problem once the basis functions $\phi_m(\mathbf{x})$ are fixed. Vary this expression with respect to $c_\beta(n,k)$; $\beta=1,2,d_L$ to find

$$\Sigma_{m=1}^{M} M(k)_{nm}\, c_\beta(m,k) = \Sigma_{r=1}^{N_B} y_\beta(r,k+1)\phi_n(y^{(r)}(k)), \tag{58}$$

where

$$M(k)_{nm} = \Sigma_{r=1}^{N_B} \phi_n(\mathbf{y}^{(r)}(k))\, \phi_m(y^{(r)}(k)). \tag{59}$$

So the problem is an $M \times M$ matrix inversion problem, and this is well studied. The literature is rife with examples of good, stable, accurate algorithms for solving this kind of problem.

When we are done with this, we will have a local model associated with each observed point $\mathbf{y}(k)$ on the attractor. In practice what one would do for prediction is to store the points $\mathbf{y}(k)$, and, when handed a point $\mathbf{z}(0)$ from which one wants to predict the evo-

lution of the system, we would search the $\mathbf{y}(k)$ to find the one nearest $\mathbf{z}(0)$, then for this point, let's call it $\mathbf{y}(J)$, we would construct the model local to its nearest neighbor among the observed points. This would give us a local model $\mathbf{F}_J(\mathbf{x})$ which should be valid as an interpolating function in the neighbourhood of $\mathbf{y}(J)$ which includes the point of interest $\mathbf{z}(0)$. Now we evaluate $\mathbf{F}_J(\mathbf{z}(0))$ and this gives us the next point on the orbit which starts with $\mathbf{z}(0)$ as initial condition: $\mathbf{z}(1)=\mathbf{F}_J(\mathbf{z}(0))$. We call this an interpolating operation because the function $\mathbf{F}_J(\mathbf{x})$ contains information in its coefficients $\mathbf{c}(m,J)$ about all the neighbors in the neighbourhood of $\mathbf{y}(J)$. Now we find the nearest neighbor of $\mathbf{z}(1)$, call it $\mathbf{y}(K)$, evaluate the required local map $\mathbf{F}_K(\mathbf{x})$ and proceed to $\mathbf{z}(2)=\mathbf{F}_K(\mathbf{z}(1))$. We iterate this procedure for as far into the future of $\mathbf{z}(0)$ as we wish to predict. The bound on our accuracy in this is determined by the error we make in the actual value of $\mathbf{z}(0)$ and the largest local Lyapunov exponent $\lambda_1(\mathbf{z}(0),L)$ which tells us how that error grows.

The procedure we have just described is called iterative forecasting since we make a large number of unit time steps to reach L steps into the future of $\mathbf{z}(0)$. The root mean square error in this forecast should scale approximately as [2,45] in going from $\mathbf{z}(0)$ to $\mathbf{z}(L)$ in L steps

$$N \cdot \langle (I+1)/d_L \rangle e^{L\lambda_1}, \tag{60}$$

if we use polynomial basis functions $\phi_n(\mathbf{x})$ where I is the maximum order of the polynomials used, and N is the number of data. If we attempt to build a model which goes from $\mathbf{z}(0)$ to $\mathbf{z}(L)$ in one direct step, then the scaling is less optimistic with the RMS error estimated to be

$$N \cdot \langle (I+1)/d_L \rangle e^{(I+1)L\lambda_1}. \tag{61}$$

4.1.1. ***Lorenz Model.*** As ever we take an example from the Lorenz model to demonstrate how these methods work. In Figure 47 we have the RMS prediction error scaled to the size of the attractor R_A for local polynomial prediction functions. The results for local linear maps are shown with circles and local quadratic maps, with squares. The error grows approximately exponentially with the number of steps ahead of any given point. The computation was done by using 48.000 data points in a reconstructed phase space d_E=3 and local maps with d_L=3. 1000 different initial conditions were examined, and the average is what we have displayed.

4.1.2. ***Chaotic Circuits.*** In Figure 48 we perform the same forecasting task as just

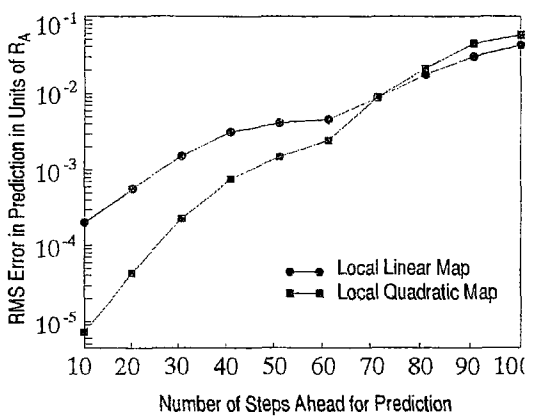

Figure 47. RMS prediction error for local linear and local quadratic map models for data from $x(n)$ from the Lorenz system. Models were built in $d_E=d_L=3$ using $T=10$ as determined from the methods described in this article. The error is in units of the size of the attractor R_A. The error grows approximately exponentialy at a rate dictated by the largest Lyapunov exponent λ_1

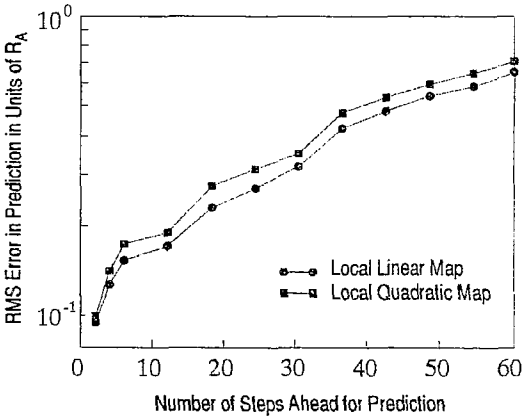

Figure 48. RMS prediction error for local linear and local quadratic map models for data from $V_B(n)$ from the hysteretic nonlinear circuit. Models were built in $d_E=d_L=3$ using $T=6$ as determined from the methods described in this article. The error is in units of the size of the attractor R_A. The error grows approximately exponentially at a rate dictated by the largest Lyapunov exponent λ_1

done for the Lorenz model but using data from the hysteretic circuit discussed several times in this article. The data was from the voltage V_B and again 48.000 data points in d_E=3 were used to make local linear and then local quadratic maps in d_L=3. The average RMS error relative to the size of the attractor is shown in the figure with an average having been taken over 1000 starting sites.

4.2. Global Models

The collection of local polynomial (or other basis function) maps form a model which is useful over the whole attractor though likely to be of less value off the attractor. The shortcomings of such a global model are its discontinuities from neighbourhood to neighbourhood and its extremely large number of adjustable parameters. For polynomial models of order M in d_L local dimensions we have approximately $d_L{}^M$ parameters at each time step. This is clearly a penalty for high accuracy. At the same time, it would be nice to have relatively simple continuous model describing the whole collection of data. A number of solely global models have been discussed which present a closed functional representation of the dynamics in the whole phase space (or, at least, on and near the whole attractor). The smoothness or analytic behavior of such models as well as the ability to produce orbits over large regions of the phase space and study their behavior as model parameters are varied makes them quite attractive.

Each method for producing global models uses some expansion of the dynamical vector field $\mathbf{F}(\mathbf{x})$ in a set of basis functions in R^d. In a sense this is precisely what we have described for local models, but we apply the idea over the whole attractor instead. The first such global method is to use polynomials again. Their advantage in local modeling where least square fitting works well, is now reduced by the extremely large number of data points and the necessity to use rather high-order polynomials.

There is an attractive alternative approach to finding a polynomial representation of a global map. This *measure-based functional reconstruction* [46,47] uses orthogonal polynomials whose weghts are determined by the invariant density on the attractor. The method eliminates the problem of multiparameter optimization. Finding the coefficients of the polynomials and the coefficients of the function $\mathbf{F}(\mathbf{x})$ requires only the evaluation of moments of data points in phase space.

The method works as follows. We introduce polynomials $\phi_a(\mathbf{x})$ on R^{d_E} which are orthogonal with respect to the natural invariant density on the attractor

$$\int d^{d_E}x\rho(\mathbf{x})\phi_m(\mathbf{x})\phi_n(\mathbf{x}) = \delta_{mn}, \tag{62}$$

and the polynomials are determined by a conventional Gram-Schmidt procedure starting from

$$\phi_1(\mathbf{x}) = 1. \tag{63}$$

The vector field $\mathbf{F}(\mathbf{x})$ which evolves data points $\mathbf{y}(k+1)=\mathbf{F}(\mathbf{y}(k))$ is approximated in M^{th} order as

$$\mathbf{F}_M(\mathbf{x}) = \Sigma_{m=1}^{M} \mathbf{c}(m)\phi_m(\mathbf{x}). \tag{64}$$

This differs from the local expansion of the same form by having the coefficients independent of the phase space location where the approximation is being made; that is, it is a global fit. The coefficients $\mathbf{c}(m)$ are determined via

$$\mathbf{c}(m) = \int d^{d_E}x\mathbf{F}(\mathbf{x})\phi_m(\mathbf{x})\rho(\mathbf{x}) = 1/N\Sigma_{k=1}^{N}\mathbf{F}(\mathbf{y}(k))\phi_m(\mathbf{y}(k)) = 1/N\Sigma_{k=1}^{N}\mathbf{y}(k+1)\phi_m(\mathbf{y}(k)). \tag{65}$$

This demonstrates the power of the method directly. Once the orthogonal polynomials $\phi_a(\mathbf{x})$ are determined from the data, the evaluation of the vector field is reduced to sums over powers of the data with themselves since the $\phi_m(\mathbf{x})$ are polynomials.

Furthermore, the form of the sums involved allow one to establish the vector field from a given set of data and adaptively improve it as new data are measured. The best aspect of the method, however, may be the robustness against contamination of the data [47]. There is no least squares parameter search involved, so no distances in state space need be evaluated. The geometric nature of the method doesn't rely on accurately determining distances and is thus not so sensitive to «noise» which spoils such distance evaluations. Further, by using the whole data set instead of just data in a local neighbourhood a certain amount of averaging and thus «noise» filtering is done automatically.

As an example of this method we show the results of its use in taking data from a map of the plane to itself and using data from one of the variables creating a global model which is then itself used to generate global Lyapunov exponents [47]. The particular issue we shall address has to do with the robustness of the global map in its prediction of the Lyapunov exponents in the presence of contaminated data. The data from the map is contaminated by gaussian random numbers of zero mean and standard deviations of varying amounts.

The map, called the Ikeda map in the literature, comes from a study of a pumped and lossy ring laser system and gives a representation of the amplitude and phase of the laser beam as it passes a plane cutting across the ring at some location. The amplitude and phase are embodied in a complex variable $z(n)=x(n)+iy(n)$ where the index n identifies the first, second, ... passage across the plane [60,61] map of the plane to itself:

$$z(n+1) = p + Bz(n) \exp [i\kappa - i\alpha/(1+|z(n)|^2)] \qquad (66)$$

where p=1.0, B=0.76, κ=0.4, α=6.0. The dimension of the attractor associated with this map is $d_A \approx 1.4$. The usual tests show that $x(n)$ data from this system can be embedded in d_E=3, and we use this information to make a global model utilizing ten orthogonal polynomials in accord to the description just provided. With this global model, the Jacobian matrix evaluated along the trajectory is used to yield Lyapunov exponents. One of the exponents is false and is identified by its behavior under time reversal. The other two exponents are approximately $\lambda_1 \approx 0.35$ and $\lambda_2 \approx -1.0$. In Figure 49 we display the two true Lyapunov exponents as a function of the amplitude of the Gaussian random number noise added to the output of the map. 1100 data points from the map are used to create the global polynomial map. The RMS level of the Ikeda map is about 0.7, so the largest contamination level which is a RMS level of 0.1 is about a 14% noise level in amplitude. In conventional signal to noise language this translates into a signal to noise ratio of 16.9 dB. The signal to noise ratio when the signal is cleanest (noise amplitude of 0.001) is 56.9 dB. The results of using the global map to evaluate the Lyapunov exponents is shown in solid symbols. Using the local polynomial method described above [38,39] gives the output shown in open symbols. It is clear that the global polynomial method, which requires substantially less computation, is much more robust against noise contamination.

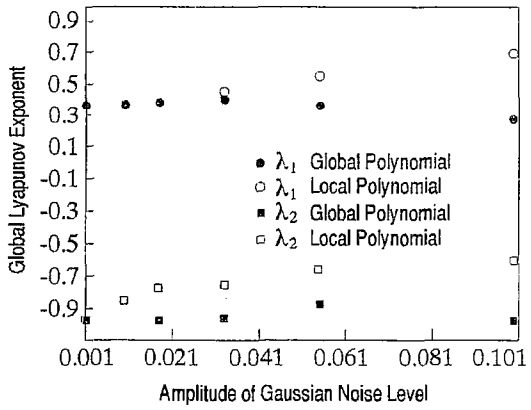

Figure 49. The global Lyapunov exponents for the Ikeda map of the plane to itself. The exponents are displayed as a function of the amplitude of Gaussian random noise added to the «observed» signal from the map. The RMS size of the signal is 0.7 in these units. The results in solid symbols comes from using a global orthogonal polynomial representation of the mapping $\mathbf{x} \rightarrow \mathbf{F}(\mathbf{x})$; the data in open circles comes from making local polynomial neighborhood to neighborhood maps as described in Section 3. The global polynomial method requires less computation and is more robust against contamination of the data

The present direction of nonlinear modeling combines features of local and global models. Consider, for example, the method of *radial basis functions* [24,44] which, as Casdagli notes [48], «is a global interpolation technique with good local-

ization properties». In this method a local predictor $\mathbf{F}(\mathbf{y})$ is sought in the form

$$\mathbf{F}(\mathbf{y}) = \Sigma_{n=1}^{N_c} \mathbf{c}(n)\Phi(\|\mathbf{y}- \mathbf{y}(n)\|) \tag{67}$$

where $\Phi(\|\mathbf{x}\|)$ is some smooth function. The coefficients $\mathbf{c}(n)$ are chosen to minimize the residuals in the usual least squares fit to the data. Depending on the number of points N_c used for reconstruction this method can be considered as local (small $N_c<<N$) or global ($N_c \approx N$).

Various choices for $\Phi(\|\mathbf{x}\|)$ will do. $\Phi(r)=(r^2+c^2)^{-\beta}$ works well for $\beta>-1$ and $\beta\neq 0$. If one adds a sum of polynomials to the sum of radial basis functions, then even increasing functions $\Phi(r)$ provide good localization properties. However, for a large number of points N_c this method is as computationally expensive as a usual least square fit. Numerical tests carried out in [48] show that for small number of data points radial basis predictors do a better job than polynomial models although for larger amount of data ($N \geq 10^4$) the local polynomial models seem to be superior.

An interesting variant of radial basis functions is kernel density estimation [49]. In this method one estimates a smooth probability distribution from discrete data points. Each point is associated with its *kernel* which is a smooth function $K(\|\mathbf{y}- \mathbf{y}(i)\|)$ which typically decays with distance, but sometimes can even increase. Using a kernel chosen *a priori* a probability distribution

$$p(\mathbf{x}) = \Sigma_i K(\|\mathbf{x}- \mathbf{y}(i)\|) \tag{68}$$

or a conditional probability distribution

$$p_c(\mathbf{x}|\mathbf{z}) = \Sigma_i K(\|\mathbf{x}- \mathbf{y}(i+1)\|)\, K(\|\mathbf{z}- \mathbf{y}(i)\|) \tag{69}$$

is estimated. $p_c(\mathbf{x}|\mathbf{z})$ can then be used for conditional forecasting by the rule for the estimated phase space point evolving from $\mathbf{y}(k)$

$$\tilde{\mathbf{y}}(k+1) = \int d\mathbf{x}\, \mathbf{x} p_c(\mathbf{x}|\mathbf{y}(k)). \tag{70}$$

Kernel density estimation usually provides the same accuracy as the first-order local predictors.

Again, computing the conditional probability distribution, one can impose weights in order to attach more value to the points close to the starting point of prediction (both in time and in phase space). In fact, this leads to a class of models which are hybrids of local and global methods. Moreover, it allows one to construct a model which possesses not only good predicting properties but also preserves important invariants of the dynamics. The prediction model by Abarbanel, Brown and Kadtke [50] belongs to this class. That model chooses the global map as

$$\mathbf{F}(\mathbf{y},\mathbf{a}) = \Sigma_{k=1}^{N-1} \mathbf{y}(k+1) g(\mathbf{y},\mathbf{y}(k);\mathbf{a}), \tag{71}$$

where $g(\mathbf{y},\mathbf{y}(k);\mathbf{a})$ is the analog of the kernel function. It is near 1 for $\mathbf{y}=\mathbf{y}(k)$ and vanishes rapidly away from there. The $\mathbf{a}$ are constants. To determine the $\mathbf{a}$ and the $\mathbf{X}$ minimize

$$\Sigma_{k=L}^{N-1} \|\mathbf{y}(k+1) - \Sigma_{n=1}^{L} X_n \mathbf{F}^n(\mathbf{y}(k- n+1),\mathbf{a})\|^2, \tag{72}$$

where X_n, $n=1,\ldots, L$ is a set of weights attached to the sequence of points $\mathbf{y}(k- n+1)$ *all* of which are to be mapped into $\mathbf{y}(k+1)$ by maps $\mathbf{F}^n(\mathbf{y})$. It is clear that then $\mathbf{F}(\mathbf{y}(k),\mathbf{a})$ will be close to $\mathbf{y}(k+1)$ as it provides an excellent interpolation function in phase space and in time. The free parameters in the kernel function $\mathbf{g}$ and the specific choice of the cost function allow one to predict forward accurately in time *and* satisfy additional constraints imposed by the dynamics-significant Lyapunov exponents and moments of the invariant

density distribution as determined by the data will be reproduced in this model. The latter does not follow automatically from the former. Indeed, it is shown in [50] that models which predict enormously accurately can have all negative Lyapunov exponents even when the data are chaotic.

4.3. A Few Remarks about Model Making

Using phase space information to establish evolution rules on the attractor has proven to be both quite easy and quite successful. If one can determine how whole neighborhoods evolve into other neighborhoods in phase space, excellent interpolation functions using a wide variety of different basis functions have been made. Though we do not dwell on their failings were we to use global linear models to capture the evolution of these systems, we would easily find that the RMS prediction error would be the same size as the attractor. It is not surprising that such a direct application of tools developed for linear systems would not succeed in situations where multivariate signals are operating and global nonlinear properties, such as positive Lyapunov exponents, and the folding of the orbit back on itself due to the dissipative nature of the system are critical elements.

The really important departure in model making is to view the system in the appropriate space, as dictated by the data. Once one does that and uses the phase space information thus exposed, many routes to successful modeling of the system present themselves.

5. Summary and Conclusions

The trajectory through the analysis of observed time series we have followed in this article is mapped out in tabular form in Table 1. It is clear that the tasks facing the analyst of a signal are essentially the same regardless of the linear or nonlinear nature of the source. The methods one adopts for the analysis are substantially different. The phase space methods used in the analysis of signals from nonlinear sources are quite usable in the case where the equations of motion governing the source are linear, but we suspect that at the present stage of development these methods do not yet compete with the well studied and carefully constructed linear Fourier space techiques available in the vast literature on this subject.

In the approach outlined in this article we have emphasized that from scalar measurements $s(n)=s(t_0+n\tau_s)$ one can directly and efficiently go to the multivariate space of vectors $\mathbf{y}(n)$ in which the system attractor is geometrically simple. Indeed, we have emphasized in an implicit way that in any smaller dimensional space, for example the one dimensional observation space of the $s(n)$ themselves, there is ambiguity associated with false neighbors which interferes with one's ability to make accurate models for prediction. The ambiguities are precisely due to orbit points which should dwell in distant reaches of phase space and thus not evolve from each other in only a few time steps finding themselves by projection nearby each other. Trying to model the evolution of these points into each other is basically an error when thought of from the point of view of the underlying multidimensional dynamics.

Once one accepts that the state space in which one must view chaotic systems is multidimensional and understands that one has to devise time domain methods for working in this space, then much of what we have presented is easier to grasp and build on. We have discussed essentially two kinds of dynamical invariants in this framework: fractal dimensions D_q and local and global Lyapunov exponents $\lambda_a(\mathbf{x},L)$; λ_a. These have a nice intuitive sense about them and have a convenient physical and geometrial interpretation. We suspect they do not provide a complete set of invariants in the sense that knowing all that one could deduce about them from the data allows one to identify the source of the signal without ambiguity. A complete set of such invariants is not known to me, but a promising route to finding such invariants has been pursued in the work of Gilmore and others [51,52,53,54,55,56,57,58] on the topology of unstable periodic orbits

Table 1

LINEAR SIGNAL PROCESSING	NONLINEAR SIGNAL PROCESSING
FINDING THE SPACE	FINDING THE SPACE
FOURIER TRANSFORMS	PHASE SPACE RECONSTRUCTION
Use Fourier space methods to turn differential equations or recursion relations into algebraic forms. $s(n)$ is observed; Fourier transform to $S(f)= \Sigma\, s(n) \exp[i2\pi nf)$	Using time lagged variables, form coordinates for the phase space in d_E dimensions: $\mathbf{y}(n)=[s(n), s(n+T),..., s(n+(d_E-1)T)]$ Determine T and d_E from average mutual information and global false nearest neighbours
CLASSIFY THE SIGNAL	CLASSIFY THE SIGNAL
Sharp Spectral Peaks Resonant Frequencies of the system Unchanged under variation in forcing while system remains *linear*	Invariants of orbits. Lyapunov Exponents; Fractal Dimensions D_q; Moments of invariant distribution function in phase space
Quantities independent of initial conditions	**Quantities independent of initial conditions**
MAKE MODELS, PREDICT, CONTROL	MAKE MODELS, PREDICT, CONTROL
$s(n+1)= \Sigma c_j s(n-j)$ Find parameters c_j by least squares method in scalar space. Make consistent with invariant classifiers (spectral peaks)	$\mathbf{y}(n) \rightarrow \mathbf{y}(n+1)$ as time evolution in d_L dimensional space determined by local false nearest neighbors $\mathbf{y}(n+1)=F[\mathbf{y}(n),a_1a_2,...a_p]$ Find parameters a_j by least squares method in multidimensional space. Make consistent with invariant classifiers (Lyapunov exponents, fractal dimensions)

which lie within all strange attractors. That work is restricted to three dimensional phase space at this time, but seems to have great promise.

The role of unstable periodic orbits in the behavior and understanding of strange attractors is a rapidly developing area of study. There is a folklore, backed up by mathematical statements about systems which do not precisely correspond to those seen in physical settings, that within all strange attractors there lies a dense set of unstable periodic orbits. The number of these orbits increases exponentially rapidly with their period, but the lowest order orbits can be identified in a given chaotic data set with some ease. The most striking utilization of these unstable orbits is the work of Ott, Grebogi and Yorke [59] where it is shown how to change the dynamical system which produces the observed chaotic orbit into a «nearby» system with a time dependent parameter. This new, but nearby, system is designed to have a stable periodic orbit very close in phase space in to the previously unstable periodic orbit. This allows one by making small alterations in the parameter to drive the system to that now stabilized orbit. Adding dimensions to the state space to provide external guiding of an orbit is more or less a standard method in control theory, but the innovative twist here is to use the stable and unstable eigendirections of the local Jacobian matrices $\mathbf{DF(x)}$ to perform the guiding for you. Since the evolution along such directions is exponentially rapid in time, small alterations in parameters can cause large changes in the behavior of the «nearby» system. There is enormous engineering potential in this technique which has only begun to be realized.

This kind of control is just one of the many topics we have not explored during our discussion here. Indeed, the number of items we have not even touched in the area of nonlinear dynamics and chaos could and does now fill numerous monographs. Some of the important topics more or less directly related to the analysis of signals includes the whole issue of signal separation, or as it is often labeled «noise reduction» [2]. This topic and the study of the behavior of chaotic signals under linear filtering operations leads one to the interesting subject of communications using chaotic signals. Viewed as a class of signals intermediate between regular sinusoidal behavior and high dimensional «noise» which is truly stochastic, it is clear that new practical opportunities abound in this domain. In the study of chaos as another class of time series, rather than the approach we have taken here which is to view it as a window into interesting dynamics of systems, one can raise a gaggle of questions traditionally explored in time series analysis. These include the study of nonstationary time series, methods for dealing with data which is not unifomly sampled in time, and data in which there are substantial gaps in the time record. All of these questions acquire renewed interest in the context of chaos and it is my hope that the tools provided in this review will allow the thorough exploration of these and other issues - all done in the multidimensional phase space in which these questions are properly posed.

This review has focused on applications and illustrations of the methods and tools in areas with which the author is familiar. This includes nonlinear circuits where we barely touched the surface of the literature on the subject. The study of chaotic hydrology exemplified by the behavior of the Great Salt Lake has immense practical importance. The data sets available in this arena are numerous, often subject to the gross uncerteinties associated with geophysical measurements, but of some real world importance. The records such as the one with which we illustrated our tools may also serve as the practical basis for evaluating the complex issue of global climate change. If one has accurate phase space models of systems which integrate in space and time such as the Great Salt Lake data, then changes in aspects of those models induced by human activity may allow definitive identification of climate evolution differing from the natural variations. In the study of chaos in laser systems we again utilized the data for illustration, but it should be clear that the practical applications of these studies are vast. We pointed to a fundamental question which absent these tools might never arise; namely, the role of spontaneous emission which acts as «intrinsic» quantum noise. With our tools we were able to distinguish between chaotic and low dimensional intensity fluctuations in these lasers and high dimensional «noise» dominated operation. The latter may arise from the spontaneous emission, but the matter is not yet settled. Finally the use of the methods to identify low dimensional aspects of high Reynolds number boundary layer flows may, coupled with control schemes of the kind alluded to just above, allow numerous practial applications to emerge from flow in pipes in power plants to flow around vehicles in all environments. In presenting these illustrations we actually had to pick and choose from an enormous set of equally interesting areas of study. The point has been to illustrate the wide applicability of the tools, and the opportunities which they provide.

The study of this kind of signal has moved from the mathematics seminar room to the workplace of the experimental scientist and the design engineer. The fruit they will bear there over the coming decades is certain to be as unexpected as it is valuable in distinct practical applications. An area where this review has not tread at all concerns the use of these methods in biological sciences, bioengineering, and medicine. I have no experience in this area, but a clear sense pervades the literature of this realm on application that there may be more here than even in the promising applications to physical and mechanical systems.

Acknowledgements

I thank the members of INLS for numerous discussions on this subject; the comments and assistance of E.Boerner, J.Goldberg, M.B.Kennel, C.Liu, M.M.Sushchik, J.J.

(«Sid») Sidorowich, and L.Sh. Tsimring were essential. I am indebted to T.Carrol, J.Cembrola, T.Galib, Z.Gills, U.Lall, L.Pecora, and R.Roy for providing me with the various data sets discussed in this review. This work was supported in part by the U.S.Department of Energy, Office of Basic Energy Sciences, Division of Engineering and Geosciences, under contract DE-FG03-90ER14138, and in part by the Army Research Office (Contract DAAL03-91-C-052), and by the Office of Naval Research (Contract N00014-91-C-012), under sub-contract to the Lockheed/Sanders Corporation.

References

[2] Abarbanel, H.D.I., R.Brown, J.J.(«SID») Sidorowich, and L.Sh.Tsimring, «The Analysis of Observed Chaotic Data in Physical Systems», to appear in Rev.Mod.Phys October, 1993.

[24] Parlitz, U., «Identification of True and Spurious Lyapunov Exponents from Time Series», em Int. J.Bif.Chaos 2,155-165(1992).

[38] Brown,R., Bryant,P. and H.D.I. Abarbanel, «Computing the Lyapunov Spectrum of a Dynamical System from Observed Time Series» Phys. Rev. A **43**, 2787 (1991; Phys. Rev. Lett. **65**, 1523 (1990).

[39] Briggs,K., «An Improved Method for Estimating Liapunov Exponents of Chaotic Time Series», Phys. Lett., **151A**, 27-32 (1990).

[43] Rissanen, J., Stochastic Complexity in Statistical Inquiry (World Scientific, Singapore), 1989.

[44] Powell, M.J.D., Approximation Theory and Methods (Cambridge University Press, Cambridge), 1981; M.J.D.Powell, 1985 «Radial basis functions for multivariate interpolation: a review» Preprint, Univ. of Cambridge; M.J.D.Powell, 1987, «Radial basis function approximation to polynomials» Preprint, Univ. of Cambridge.

[45] Farmer, J.D. and J.J. («Sid») Sidorowich in Evolution, Learning and Codnition ed. Y.-C.Lee (World Scientific, Singapore), pp 177-, 1988.

[46] Giona,M.,F. Lentini, and V.Cimagalli, «Functional Reconstruction and Local Prediction of Chaotic Time Series», Phys.Rev.A,**44** 3496-3502 (1991).

[47] Brown,R. «Orthogonal Polynomials as Prediction Functions in Arbitrary Phase Space Dimensions», Phys. Rev. E, **47** 3962-3969, (1993).

[48] Casdagli, M., «Nonlinear Prediction of Chaotic Time Series», Physica D **20** 335-356, (1989).

[49] Silverman, B.W. Density Estimation of Statistics and Data Analysis, Chapman and Hall, London), 1986.

[50] Abarbanel, H.D.I,R. Brown, and J.B.Kadtke, «Prediction in Chaotic Nonlinear Systems: Methods for Time Series with Broadband Fourier Spectra», Phys.Rev.A **41**,1782-1807 (1990).

[51] Mindlin, G.B.,H.G.Solari, M.A.Natiello, R.Gilmore, and X.-J.Hou, «Topological Analysis of Chaotic Time Series Data from the Belousov-Zhabotinskii Reaction», J.Nonlin.Sci. **1**, 147-173 (1991)

[52] Birman, J.S.and R.F.Williams, «Knotted Periodic Orbits in Dynamical Systems I: Lorenz`s Equations», Topology **22**, 47-82 (1983).

[53] Birman, J.S. and R.F.Williams, «Knotted Periodic Orbits in Dynamical Systems II: Knot Holders for Fibered Knots», Cont.Math.**20**,1-60 (1983).

[54] Papoff, F.,A. Fioretti, E.Arimondo, G.B.Mindlin, H.Solari, and R.Gilmore, «Structure of Chaos in the Laser with Saturatable Absorber», Phys.Rev.Lett. **68**, 1128-1131 (1992).

[55] Cvitanovic,P., «Invariant Measurement of Strange Sets in Terms of Cycles», Phys. Rev. Lett. **61**, 2729-2732 (1988).

[56] «Classification of Strange Attractors by Integers», Mindlin, G.B.,X-J Hou, H.G.Solari, R.Gilmore, and N.B.Tufillaro, Phys.Rev.Lett. **64**,2350-2353 (1990).

[57] Flepp, L.,R.Holner, E.Brun, M.Finardi, and R.Badii, «Model Identification by Periodic-Orbit Analysis for NMR-Laser Chaos», Phys.Rev.Lett. **67**, 2244-2247 (1991).

[58] Tufillaro, N.B., R.Holzner, L.Flepp, E.Brun, M.Finardi, and R.Badii, «Template Analysis for a Chaotic NMR Laser», Phys.Rev.A.**44** R4786-R4788 (1991).

[59] Ott,E.,C.Grebogi, and J.A.Yorke, «Controlling Chaos», Phys.Rev.Lett. 1196-1199 (1990). See also their review article in Nature **363**,411-417 (1993) where an extensive list of references to theoretical and experimental work is given.

[60] Ikeda, K., «Multiple-Valued Stationary State and Its Instability of the Transmitted Light by a Ring Cavity System», Opt.Commun. **30**, 257 (1979).

[61] Hammel, S.M., Jones, C.K.R.T. and Moloney, J.V., J.Opt.Soc.Am. **B 2**, 552 (1985).

Institute for Nonlinear Science

Поступила в редакцию 18.12.94

Генри Д.И. Абарбанель - профессор физики и физик-исследователь Института океанографии Скриппса в Калифорнийском университете, Сан-Диего, UCSD, США. Получил степень доктора в Принстонском университете и преподавал в Принстоне, Станфорде, Нортвестерне, Чикаго и Калифорнийском университете в Беркли, с 1983 года работает в Калифорнийском университете в Сан-Диего. Занимается анализом хаотических явлений в геофизических и лабораторных установках. Директор Института нелинейных наук при UCSD.